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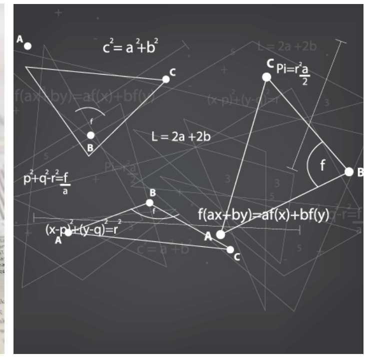

Triangles and Triplets
Heron meets Pythagoras

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Let's be honest. If you have a triangular room whose floor needs tiling you are not going to find area using one-half base times height. For starters, how will you determine where the altitude intersects the base? This is easy if dealing with a right triangle but not if your room is in the shape of a "wrong" one. Being realistic, the only useful measurements are of the room's three sides. To get the job done, what is needed is a formula finding area using side measurements and not base and height. Fortunately, such a formula exists—Heron's formula.

For a triangle of sides a, b, and c, we define semi-perimeter, S, as half the perimeter, i.e. $S=½(a + b + c)$. Area is then found using: $A^2=S(S–a)(S–b)(S–c)$. Hence, if our floor had sides of lengths: 14, 20, and, 28 then $s = 31$ and $A^2 = 31x17x11x3=17391$ our area is 132 rounded up. But, are we sure?

The proof of Heron's formula can be readily found online, but it's not a pretty sight. So we're going to take a different approach—one popular for a short time in Newton's England—we'll find a way to demonstrate it with examples. Such an exercise requires that we come up with an obtuse triangle whose sides and altitude are all integer values, which in turn, requires coming-up with numerous Pythagorean Triplets. (Three integers A, B, & C where $A^2 + B^2 = C^2$). Our plan is simple: find two triplets sharing a common number and use that number as the triangle's altitude. That is, we create our obtuse triangle by abutting two right triangles that share a common side. To this end, we use a little known rule for generating Pythagorean Triplets.

We begin by defining bi-consecutive numbers as two integers differing by 2. Pairs 6 & 8 are bi-consecutive as are 13 & 15. In general, we let m and n be bi-consecutive. It turns-out that (m + n) and (m x n) are always the A and B of a Pythagorean triplet. Let's demonstrate:
$6 + 8 = 14$ and $6 \times 8 = 48 \rightarrow 14^2 + 48^2 = 196 + 2304 = 2500 = 50^2$. Or, if one example is not good enough, $13 + 15 = 28$ and $13 \times 15 = 195 \rightarrow 28^2 + 195^2 = 784 + 38025 = 38809 = 197^2$. We will delay a proof of this result until later.

Returning to our plan, we're looking for a number that can be expressed as the sum of two bi-consecutive numbers, as well as the product of two other bi-consecutive numbers. As the product of bi-consecutive numbers is always of the form $N^2 – 1$* and as the sum of two bi-consecutive numbers is always even, we seek an even number one less than a perfect square. For example, 48. We express this number as the sum of two bi-consecutive numbers, in this case 23 and 25, and as the product of two bi-consecutive numbers 6 x 8. This makes 48 part of two Pythagorean triplets: $48^2 + 14^2$ (found earlier) and $48^2 + 575^2 = 2304 + 330625 = 332929 = 577^2$. However, there are a lot more: $24 = 25 – 1$, $80 = 81 – 1$, and so on and so on.

Using $½\, h \cdot b$ gives us $A = 28272 \div 2 = 14136$. Using Heron's formula, we calculate $S= ½\,(50 + 577 + 589) = 608$. Thus $A^2 = 608 \times 558 \times 31 \times 19 = 14136^2$ and we are done!

Proof of Rule: If a and b are bi-consecutive then for $n = (a + b)/2$, $a = n – 1$ and $b = n + 1$.
Letting $A = a \times b = (n – 1)(n + 1) = n^2 – 1$ [refer back to *], and $B = a + b = 2n$, we get:
$A^2 + B^2 = (n^2 – 1)^2 + 4n^2 = (n^4 – 2n^2 + 1) + 4n^2 = n^4 + 2n^2 + 1 = (n + 1)^2$. We point out that this result can be easily extended to cover ***all*** Pythagorean triplets by replacing $n^2 – 1$ by $n^2 – m^2$, and 2n by 2nm and let n and m be any positive integer.

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